CHAPTER 1
INTRODUCTION TO MODERN NETWORK THEORY

1.1 MODERN NETWORK THEORY

A generalized filter is shown in Figure 1-1. The filter block may consist of inductors, capacitors, resistors, and possibly active elements such as operational amplifiers and transistors. The terminations shown are a voltage source E_s, a source resistance R_s, and a load resistor R_L.

The circuit equations for the network of Figure 1-1 can be written by using circuit-analysis techniques. Modern network theory solves these equations to determine the network values for optimum performance in some respect.

The Pole-Zero Concept

The frequency response of the generalized filter can be expressed as a ratio of two polynomials in s where $s = j\omega$ ($j = \sqrt{-1}$, and ω, the frequency in radians per second, is $2\pi f$) and is referred to as a transfer function. This can be stated mathematically as

$$T(s) = \frac{E_L}{E_s} = \frac{N(s)}{D(s)} \tag{1-1}$$

The roots of the denominator polynomial $D(s)$ are called poles and the roots of the numerator polynomial $N(s)$ are referred to as zeros.

Deriving a network's transfer function could become quite tedious and is beyond the scope of this book. The following discussion explores the evaluation and representation of a relatively simple transfer function.

Analysis of the low-pass filter of Figure 1-2*a* results in the following transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

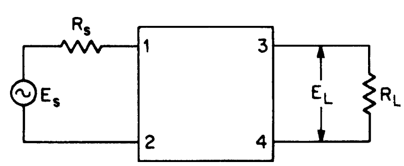

FIGURE 1-1 A generalized filter.

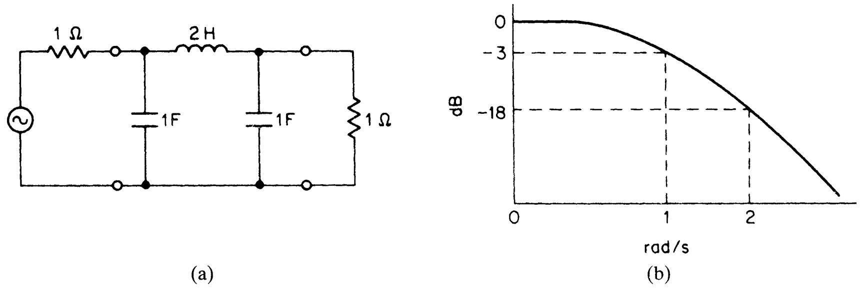

FIGURE 1-2 An all-pole $n = 3$ low-pass filter: (*a*) a filter circuit; and (*b*) a frequency response.

Let us now evaluate this expression at different frequencies after substituting $j\omega$ for s. The result will be expressed as the absolute magnitude of $T(j\omega)$ and the relative attention in decibels with respect to the response at DC.

$$T(j\omega) = \frac{1}{1 - 2\omega^2 + j(2\omega - \omega^3)} \tag{1-3}$$

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0	1	0 dB
1	0.707	−3 dB
2	0.124	−18 dB
3	0.0370	−29 dB
4	0.0156	−36 dB

The frequency-response curve is plotted in Figure 1-2*b*.

Analysis of Equation (1-2) indicates that the denominator of the transfer function has three roots or poles and the numerator has none. The filter is therefore called an all-pole type. Since the denominator is a third-order polynomial, the filter is also said to have an $n = 3$ complexity. The denominator poles are $s = -1$, $s = -0.500 + j0.866$, and $s = -0.500 - j0.866$.

These complex numbers can be represented as symbols on a complex-number plane. The abscissa is α, the real component of the root, and the ordinate is β, the imaginary part. Each pole is represented as the symbol *X*, and a zero is represented as 0. Figure 1-3 illustrates the complex-number plane representation for the roots of Equation (1-2).

Certain mathematical restrictions must be applied regarding the location of poles and zeros in order for the filter to be realizable. They must occur in pairs which are conjugates of each other, except for real-axis poles and zeros, which may occur singly. Poles must also be restricted to the left plane (in other words, the real coordinate of the pole must be negative), while zeros may occur in either plane.

Synthesis of Filters from Polynomials. Modern network theory has produced families of standard transfer functions that provide optimum filter performance in some desired respect. Synthesis is the process of deriving circuit component values from these transfer functions. Chapter 11 contains extensive tables of transfer functions and their associated component values so that design by synthesis is not required. Also, computer programs on the CD-ROM simplify the design process. However, in order to gain some understanding

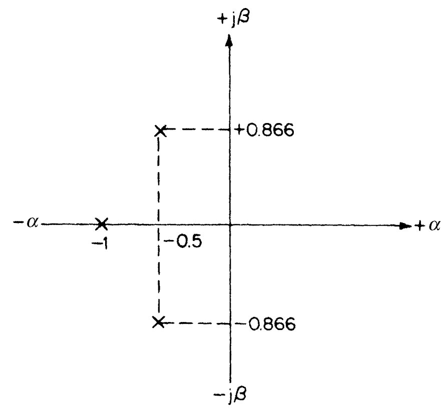

FIGURE 1-3 A complex-frequency plane representation of Equation (1-2).

as to how these values have been determined, we will now discuss a few methods of filter synthesis.

Synthesis by Expansion of Driving-Point Impedance. The input impedance to the generalized filter of Figure 1-1 is the impedance seen looking into terminals 1 and 2 with terminals 3 and 4 terminated, and is referred to as the driving-point impedance or Z_{11} of the network. If an expression for Z_{11} could be determined from the given transfer function, this expression could then be expanded to define the filter.

A family of transfer functions describing the flattest possible shape and a monotonically increasing attenuation in the stopband is known as the *Butterworth low-pass response*. These all-pole transfer functions have denominator polynomial roots, which fall on a circle having a radius of unity from the origin of the $j\omega$ axis. The attenuation for this family is 3 dB at 1 rad/s.

The transfer function of Equation (1-2) satisfies this criterion. It is evident from Figure 1-3 that if a circle were drawn having a radius of 1, with the origin as the center, it would intersect the real root and both complex roots.

If R_s in the generalized filter of Figure 1-1 is set to 1 Ω, a driving-point impedance expression can be derived in terms of the Butterworth transfer function as

$$Z_{11} = \frac{D(s) - s^n}{D(s) + s^n} \tag{1-4}$$

where $D(s)$ is the denominator polynomial of the transfer function and n is the order of the polynomial.

After $D(s)$ is substituted into Equation (1-4), Z_{11} is expanded using the continued fraction expansion. This expansion involves successive division and inversion of a ratio of two polynomials. The final form contains a sequence of terms, each alternately representing a capacitor and an inductor and finally the resistive termination. This procedure is demonstrated by the following example.

Example 1-1 Synthesis of $N = 3$ Butterworth Low-Pass Filter by Continued Fraction Expansion

Required:

A low-pass *LC* filter having a Butterworth $n = 3$ response.

Result:

(*a*) Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

(*b*) Substitute $D(s) = s^3 + 2s^2 + 2s + 1$ and $s^n = s^3$ into Equation (1-4), which results in

$$Z_{11} = \frac{2s^2 + 2s + 1}{2s^3 + 2s^2 + 2s + 1} \tag{1-4}$$

(*c*) Express Z_{11} so that the denominator is a ratio of the higher-order to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{2s^3 + 2s^2 + 2s + 1}{2s^2 + 2s + 1}}$$

(*d*) Dividing the denominator and inverting the remainder results in

$$Z_{11} = \frac{1}{s + \dfrac{1}{\dfrac{2s^2 + 2s + 1}{s + 1}}}$$

(*e*) After further division and inversion, we get as our final expression:

$$Z_{11} = \frac{1}{s + \dfrac{1}{2s + \dfrac{1}{s + 1}}} \tag{1-5}$$

The circuit configuration of Figure 1-4 is called a ladder network, since it consists of alternating series and shunt branches. The input impedance can be expressed as the following continued fraction:

$$Z_{11} = \frac{1}{Y_1 + \dfrac{1}{Z_2 + \dfrac{1}{Y_3 + \cdots \dfrac{1}{Z_{n-1} + \dfrac{1}{Y_n}}}}} \tag{1-6}$$

where $Y = sC$ and $Z = sL$ for the low-pass all-pole ladder except for a resistive termination where $Y_n = sC + 1/R_L$.

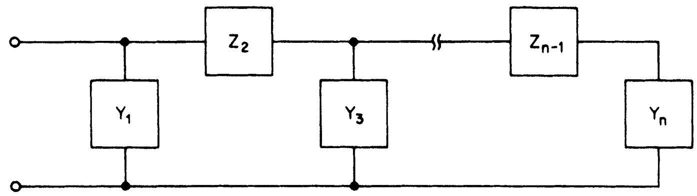

FIGURE 1-4 A general ladder network.

Figure 1-5 can then be derived from Equation (1-5) and (1-6) by inspection. This can be proved by reversing the process of expanding Z_{11}. By alternately adding admittances and impedances while working toward the input, Z_{11} is verified as being equal to Equation (1-5).

Synthesis for Unequal Terminations. If the source resistor is set equal to 1 Ω and the load resistor is desired to be infinite (unterminated), the impedance looking into terminals 1 and 2 of the generalized filter of Figure 1-1 can be expressed as

$$Z_{11}\frac{D(s\text{ even})}{D(s\text{ odd})} \tag{1-7}$$

$D(s$ even) contains all the even-power s terms of the denominator polynomial and $D(s$ odd) consist of all the odd-power s terms of any realizable all-pole low-pass transfer function. Z_{11} is expanded into a continued fraction, as in Example 1-1, to define the circuit.

Example 1-2 Synthesis of $N = 3$ Butterworth Low-Pass Filter for an Infinite Termination

Required:
Low-pass filter having a Butterworth $n = 3$ response with a source resistance of 1 Ω and an infinite termination.

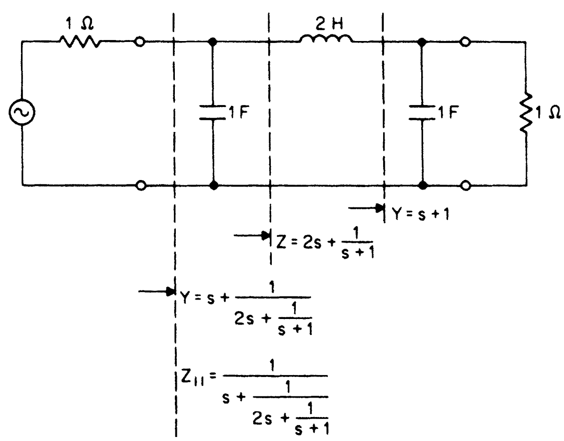

FIGURE 1-5 The low-pass filter for Equation (1-5).

Result:

(*a*) Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

(*b*) Substitute $D(s \text{ even}) = 2s^2 + 1$ and $D(s \text{ odd}) = s^3 + 2s$ into Equation (1-7):

$$Z_{11} = \frac{2s^2 + 1}{s^3 + 2s} \tag{1-7}$$

(*c*) Express Z_{11} so that the denominator is a ratio of the higher- to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{s^3 + 2s}{2s^2 + 1}}$$

(*d*) Dividing the denominator and inverting the remainder results in

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{\dfrac{2s^2 + 1}{1.5s}}}$$

(*e*) Dividing and further inverting results in the final continued fraction:

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{1.333s + \dfrac{1}{1.5s}}} \tag{1-8}$$

The circuit is shown in Figure 1-6.

Synthesis by Equating Coefficients. An active three-pole low-pass filter is shown in Figure 1-7. Its transfer function is given by

$$T(s) = \frac{1}{s^3A + s^2B + sC + 1} \tag{1-9}$$

where $$A = C_1C_2C_3 \tag{1-10}$$

$$B = 2C_3(C_1 + C_2) \tag{1-11}$$

and $$C = C_2 + 3C_3 \tag{1-12}$$

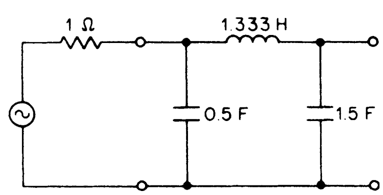

FIGURE 1-6 The low-pass filter of Example 1-2.

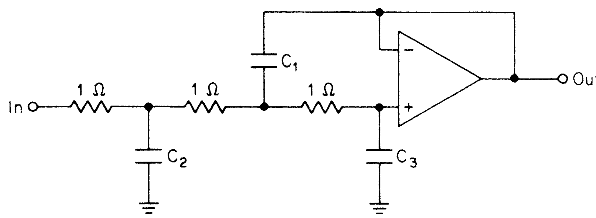

FIGURE 1-7 The general $n = 3$ active low-pass filter.

If a Butterworth transfer function is desired, we can set Equation (1-9) equal to Equation (1-2).

$$T(s) = \frac{1}{s^3A + s^2B + sC + 1} = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-13}$$

By equating coefficients, we obtain

$$A = 1$$
$$B = 2$$
$$C = 2$$

Substituting these coefficients in Equation (1-10) through (1-12) and solving for C_1, C_2, and C_3 results in the circuit of Figure 1-8.

Synthesis of filters directly from polynomials offers an elegant solution to filter design. However, it also may involve laborious computations to determine circuit element values. Design methods have been greatly simplified by the curves, tables, computer programs, and step-by-step procedures provided in this handbook, so design by synthesis can be left to the advanced specialist.

Active vs. Passive Filters. The LC filters of Figures 1-5 and 1-6 and the active filter of Figure 1-8 all satisfy an $n = 3$ Butterworth low-pass transfer function. The filter designer is frequently faced with the sometimes difficult decision of choosing whether to use an active or LC design. A number of factors must be considered. Some of the limitations and considerations for each filter type will now be discussed.

Frequency Limitations. At subaudio frequencies, LC filter designs require high values of inductance and capacitance along with their associated bulk. Active filters are more practical because they can be designed at higher impedance levels so that capacitor magnitudes are reduced.

Above 20 MHz or so, most commercial-grade operational amplifiers have insufficient open-loop gain for the average active filter requirement. However, amplifiers are available

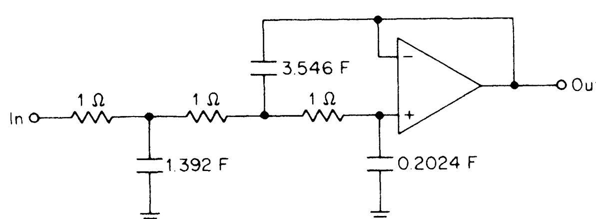

FIGURE 1-8 A Butterworth $n = 3$ active low-pass filter.

with extended bandwidth at an increased cost so that active filters at frequencies up to 100 MHz are possible. LC filters, on the other hand, are practical at frequencies up to a few hundred megahertz. Beyond this range, filters become impractical to build in lumped form, and so distributed parameter techniques are used, such as stripline or microstrip, where a PC board functions as a distributed transmission line.

Size Considerations. Active filters are generally smaller than their LC counterparts since inductors are not required. Further reduction in size is possible with microelectronic technology. Surface mount components for the most part have replaced Hybrid technology, whereas in the past Hybrids were the only way to reduce the size of active filters.

Economics and Ease of Manufacture. LC filters generally cost more than active filters because they use inductors. High-quality coils require efficient magnetic cores. Sometimes, special coil-winding methods are needed as well. These factors lead to the increased cost of LC filters.

Active filters have the distinct advantage that they can be easily assembled using standard off-the-shelf components. LC filters require coil-winding and coil-assembly skills. In addition, eliminating inductors prevents magnetic emissions, which can be troublesome.

Ease of Adjustment. In critical LC filters, tuned circuits require adjustment to specific resonances. Capacitors cannot be made variable unless they are below a few hundred picofarads. Inductors, however, can easily be adjusted, since most coil structures provide a means for tuning, such as an adjustment slug for a Ferrite potcore.

Many active filter circuits are not easily adjustable, however. They may contain RC sections where two or more resistors in each section have to be varied in order to control resonance. These types of circuit configurations are avoided. The active filter design techniques presented in this handbook include convenient methods for adjusting resonances where required, such as for narrowband bandpass filters.

BIBLIOGRAPHY

Guillemin, E. A. (1957). *Introduction to Circuit Theory*. New York: John Wiley and Sons.

Stewart, J. L. (1956). *Circuit Theory and Design*. New York: John Wiley and Sons.

White Electromagnetics. (1963). *A Handbook on Electrical Filters*. White Electromagnetics, Inc.